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NOTE ON: N. E. AGUILERA, M. S. ESCALANTE, G. L. NASINI, "A GENERALIZATION OF THE PERFECT GRAPH THEOREM UNDER THE DISJUNCTIVE INDEX"

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We give short elementary proofs of two results by N. E. Aguilera, M. S. Escalante, and G. L. Nasini on the disjunctive index of the clique relaxation of the stable set polytope.

We give short elementary proofs of two results by Aguilera et al. (2002), namely (1) and (2) below.

In this note $G = (V, E)$ is a graph, $\overline{G}$ its complement, $S(G)$ its stable set polytope, $K(G)$ the set of all $x \in \mathbb{R}_+^V$ satisfying $\sum_{u\in U} x_u \leq 1$ for each clique $U \subseteq V$ of G. If $x \in \mathbb{R}^V$ and $J \subseteq V$, then x_J is the restriction of x to J, $G[J]$ the subgraph of G induced by J and $G - J = G[V\backslash J]$. If Q is a polyhedron in $\mathbb{R}_+^V$, then $P_J Q := \text{convex hull}\{x \in Q \mid x_J \in \mathbb{Z}^J\}$ and $AQ := \{y \in \mathbb{R}_+^V \mid x^T y \leq 1 \ (x \in Q)\}$. By definition, $AK(G) = S(\overline{G})$. Hence, by standard polyhedral theory, $AS(\overline{G}) = AAK(G) = K(G)$.

The main result of Aguilera et al. (2002) is, for $J \subseteq V$,

$$\text{(1)} \qquad \text{If } P_J K(G) = S(G) \quad \text{then} \quad P_J K(\overline{G}) = (\overline{G}).$$

Here is an easy proof. Each vertex of $P_J K(G)$ is for some $S \subseteq J$ a vertex of $F_S(G) := \{x \in K(G) \mid x_S = 1;\ x_{J\backslash S} = 0\}$ and conversely . Each such $F_S(G)$ is the translation of a face of $F_\emptyset(G) = \{x \in \mathbb{R}_+^V \mid x_J = 0;\ x_{V\backslash J} \in K(G - J)\}$ over the characteristic vector of S. So $P_J K(G) = S(G)$ if and only if $K(G - J) = S(G - J)$. Because $AK(G - J) = S(\overline{G} - J)$ and $AS(G - J) = K(\overline{G} - J)$, this proves (1).

In Aguilera et al. (2002), (1) is derived from the following result:

$$\text{(2)} \qquad P_J A P_J K(G) = AK(G).$$

This implies (1) because if (2) and $P_J K(G) = S(G)$, then $P_J K(\overline{G}) = P_J AS(G) = P_J A P_J K(G) = AK(G) = S(\overline{G})$.

The derivation of (2) in Aguilera et al. (2002) is quite technical. Here is an easy proof. As $P_J A$ is decreasing with respect to inclusion, $P_J A P_J K(G) \supseteq P_J AK(G) = P_J S(\overline{G}) = S(\overline{G}) = AK(G)$.

To see the reverse inclusion, let x be a vertex of $P_J A P_J K(G)$. So x_J is a $\{0, 1\}$-vector and $x \in AP_J K(G)$. Each node u with $x_u = 1$ is adjacent to each node $v \neq u$ with $x_v > 0$, because $x \in AP_J K(G) \subseteq AS(G)$ and $x_u + x_v > 1$. Hence, assuming $x \notin AK(G) = S(\overline{G})$ and defining $U := \{v \mid 0 < x_v < 1\}$, we know $x_U \notin S(\overline{G}[U]) = AK(G[U])$. So $x_U^T y' > 1$ for some $y' \in K(G[U])$. Extending y' by 0's gives $y \in P_J K(G)$ with $x^T y > 1$. This contradicts $x \in AP_J K(G)$.

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